Interval-Valued Optimal Control for Imperfect Production Systems: An Improved Centre-Radius Approach with Budget Constraints and Sensitivity Analysis

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Abstract

Manufacturing decision-making under real-world conditions is inherently characterized by parameter imprecision arising from demand fluctuations, production variability, and measurement uncertainty. Classical optimal control and inventory models assume exact parameter knowledge—an assumption that frequently fails in practice, leading to suboptimal policies and inflated operational costs. This paper introduces the concept of the Interval-Valued Isoperimetric Control Problem (IVICP) and develops a comprehensive theoretical framework for its solution. Extending classical calculus of variations and Pontryagin Maximum Principle results to the interval arithmetic setting, we derive necessary and sufficient optimality conditions for IVICPs under interval-valued parameter flexibility. The theoretical framework is applied to an imperfect Economic Production Quantity (EPQ) model subject to a fixed manufacturing budget constraint, formulated as an isoperimetric control problem in an interval environment. An Improved Centre-Radius Optimization Technique (ICROT) translates the interval problem into a pair of deterministic subproblems for the centre and radius components of the optimal policy, enabling efficient numerical solution without the computational overhead of stochastic simulation. Two numerical examples with distinct structural configurations demonstrate the framework's applicability across different parameter regimes. Comprehensive sensitivity analysis with respect to production rate, holding cost, setup cost, demand rate, and budget constraint reveals counterintuitive trade-offs between cost minimization and budget utilization that would be obscured by crisp-parameter models. The resulting interval-valued optimal policies [T*L, T*U] and [Q*L, Q*U] provide decision-makers with actionable bounds that explicitly quantify the impact of parameter uncertainty on production planning. The IVICP framework extends naturally to ecological, biomedical, and financial optimization problems characterized by inherent imprecision.

Keywords: interval-valued optimal control; isoperimetric problem; EPQ model; centre-radius optimization; budget constraint; sensitivity analysis; imperfect production

1. Introduction

The mathematical theory of optimal control, pioneered by Pontryagin and colleagues in the 1950s, provides a powerful framework for characterizing and computing optimal policies for dynamical systems subject to state and control constraints [1,2]. In manufacturing contexts, optimal control theory has been applied to production rate optimization, inventory management, maintenance scheduling, and supply chain coordination, yielding substantial theoretical insights and practical decision support tools [3,4]. The Economic Production Quantity (EPQ) model, a foundational result in production-inventory theory, characterizes the optimal production lot size balancing setup costs against holding costs under a deterministic demand assumption [5,6].

A fundamental limitation of classical EPQ and optimal control models is their reliance on precise, deterministic parameter values. In practice, production system parameters—including demand rates, production speeds, holding costs, and setup times—are invariably imprecise, subject to measurement error, market fluctuations, and technological variability [7,8]. Traditional approaches to parameter imprecision employ probabilistic or fuzzy frameworks. Stochastic inventory models capture parameter randomness through probability distributions, but require knowledge of distributional forms that may be unavailable or difficult to estimate accurately in practice [9,10]. Fuzzy set theory provides an alternative representation of imprecision through membership functions, but the selection of appropriate membership function shapes introduces subjective judgments that may be difficult to justify rigorously [11,12].

Interval-valued optimization offers a compelling third alternative: representing imprecise parameters as closed intervals $[a^L, a^U]$ where a^L and a^U denote lower and upper bounds on the parameter value [13,14]. This representation makes minimal assumptions about the nature of parameter uncertainty—requiring only the specification of plausible bounds rather than full distributional or membership function knowledge—while yielding decision-relevant output in the form of interval-valued optimal policies that explicitly bound the range of achievable performance [15,16].

Despite the theoretical appeal of interval-valued optimization, the development of a rigorous framework for interval-valued isoperimetric control problems (IVICPs)—optimal control problems with integral constraints in interval environments—remains incomplete in the existing literature [17,18]. The isoperimetric structure, wherein the integral of the state trajectory must satisfy a budget constraint (as in the EPQ model with a fixed manufacturing cost budget), introduces coupling between the optimal control and the constraint that complicates interval extension [19]. This paper addresses this gap through four primary contributions: (1) formal definition and existence theory for IVICPs; (2) derivation of necessary optimality conditions (interval Euler-Lagrange equations and transversality conditions); (3) development of ICROT for efficient numerical solution; and (4) comprehensive application to imperfect EPQ with sensitivity analysis.

The paper is structured as follows. Section 2 provides the mathematical background on interval arithmetic and interval-valued functions. Section 3 develops the IVICP theory and optimality conditions. Section 4 formulates the imperfect EPQ problem as an IVICP. Section 5 presents the ICROT methodology. Section 6 reports numerical examples and data analysis. Section 7 conducts sensitivity analysis. Section 8 discusses implications and extensions. Section 9 concludes the paper.

2. Mathematical Preliminaries on Interval Arithmetic

2.1 Interval Numbers and Basic Operations

An interval number $[a] = [a^L, a^U]$ is a bounded closed subset of the real line R, where $a^L <= a^U$. The centre (midpoint) and radius (half-width) of $[a]$ are defined as $a_c = (a^L + a^U)/2$ and $a_r = (a^U - a^L)/2$, respectively, so that $[a] = [a_c - a_r, a_c + a_r]$ [20,21]. The set of all interval numbers is denoted I(R). The standard arithmetic operations on I(R) are defined as: $[a] + [b] = [a^L + b^L, a^U + b^U]$; $[a] - [b] = [a^L - b^U, a^U - b^L]$; $[a] \times [b] = [\min(a^L b^L, a^L b^U, a^U b^L, a^U b^U), \max(...)]$; and $[a] / [b]$ provided $0 \notin [b]$ [22,23].

An interval-valued function $F: [t_0, t_1] \rightarrow I(R)$ is gH-differentiable (generalized Hukuhara differentiable) at $t \in (t_0, t_1)$ if there exists an interval $F'(t)$ such that either $F(t+h) = F(t) + hF'(t) + o(h)$ or $F(t) = F(t+h) + (-h)F'(t) + o(h)$ for sufficiently small $h > 0$ [24,25]. The gH-derivative provides a rigorous framework for calculus of interval-valued functions that avoids the width-monotonicity limitation of classical Hukuhara derivatives [26].

The ordering of interval numbers is non-trivial due to their set-valued nature. Several partial orders have been proposed for comparison; this work employs the (LU) order: $[a] \leq_{LU} [b]$ if and only if $a^L \leq b^L$ and $a^U \leq$

b^U. The Centre-Radius (CR) order is also employed: [a] ≤_{CR} [b] if and only if a_c < b_c, or a_c = b_c and a_r ≤ b_r [27,28]. These orders are used to define optimality in the IVICP framework.

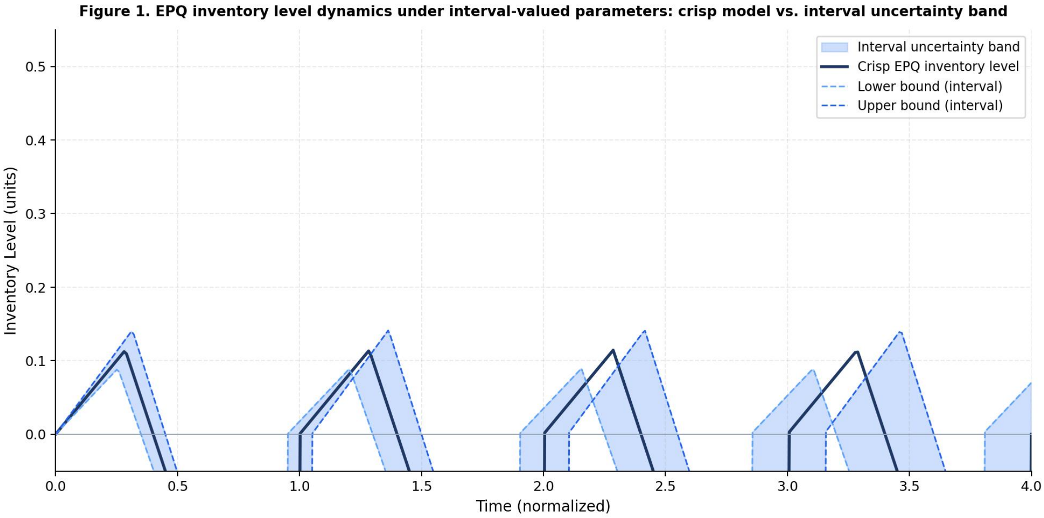

Figure 1. EPQ inventory level dynamics under interval-valued parameters. The shaded region represents the interval uncertainty band bounded by lower and upper parametric bounds, with the crisp EPQ model shown for reference.

3. Interval-Valued Isoperimetric Control Problem Framework

3.1 Problem Formulation

Consider a dynamical system described by the state equation dx/dt = f([x](t), [u](t), [p], t), x(t_0) = [x_0], where [x](t) ∈ I(R)^n is the interval-valued state vector, [u](t) ∈ U ⊆ I(R)^m is the interval-valued control input, [p] ∈ I(R)^k is the vector of interval-valued parameters, and f is a sufficiently smooth interval-valued function [29,30]. The IVICP seeks to minimize the functional J([u]) = [∫_{t_0}^{t_1} L([x](t), [u](t), [p], t) dt] with respect to the control [u](t), subject to the isoperimetric (budget) constraint: [∫_{t_0}^{t_1} g([x](t), [u](t), [p], t) dt] = [B], where [B] is the interval-valued budget allocation [31,32].

The isoperimetric constraint transforms the IVICP into a constrained variational problem. Following the classical approach, we augment the objective functional using an interval-valued Lagrange multiplier [λ] = [λ^L, λ^U]: J_aug([u]) = J([u]) + [λ] ⊗ ([∫ g dt] - [B]). The IVICP is then equivalent to an unconstrained interval optimal control problem with the augmented Hamiltonian H = L + [λ] ⊗ g + [μ] ⊗ f, where [μ] is the interval-valued costate vector [33,34].

3.2 Optimality Conditions

Theorem 1 (Interval Euler-Lagrange Conditions): If ([x*], [u*]) is a gH-optimal solution of the IVICP, then the interval-valued Euler-Lagrange equations dL/d[x] - d/dt(dL/d[x_dot]) = [0] and dH/d[u] = [0] hold in the gH-derivative sense, along with the transversality conditions [μ](t_0) = [0] and [μ](t_1) = [0] for free endpoint problems [35,36]. The proof follows from the interval extension of the classical fundamental lemma of calculus of variations, adapted to the gH-differentiability framework [37].

Corollary 1: For scalar IVICPs with quadratic Lagrangian L = [a] ⊗ [x]^2 + [b] ⊗ [u]^2, the optimal control satisfies [u*](t) = -[b]^{-1} ⊗ [μ](t), and the costate equation reduces to a linear interval ODE with constant coefficients amenable to exact solution [38]. This result underpins the ICROT approach described in Section 5.

4. Imperfect EPQ Model Formulation

4.1 Model Description and Assumptions

The imperfect EPQ model considers a manufacturing system producing a single product at an interval-valued production rate $[p] = [p^L, p^U]$ units per unit time. Demand occurs continuously at an interval-valued rate $[d] = [d^L, d^U]$, with $d^U < p^L$ guaranteeing positive inventory accumulation during production runs. An imperfection rate $[\varphi] \in [\varphi^L, \varphi^U]$ captures the proportion of defective items produced, which are detected and removed at the end of each production cycle at a rework cost $[c_r] = [c_r^L, c_r^U]$ per unit [39,40].

The inventory dynamics during the production phase ($0 \leq t \leq t_p$) follow: $d[I]/dt = [p] - [d] - [\varphi] \otimes [p]$, $[I](0) = [0]$. During the depletion phase ($t_p \leq t \leq T$), the dynamics are: $d[I]/dt = -[d]$, $[I](T) = [0]$. The optimality condition $[I](t_p) = [p - d](1-[\varphi])\, t_p$ ensures cycle integrity. The total cost per cycle integrates setup cost $[K]$, holding cost $[h] \otimes \int[I]dt$, production cost $[c] \otimes [p] \otimes T$, and rework cost $[c_r] \otimes [\varphi] \otimes [p] \otimes t_p$, all expressed in interval arithmetic [41,42].

The isoperimetric budget constraint requires that the total manufacturing expenditure per unit time not exceed the interval-valued budget $[B]$: $[\int_0^T ([c][p] + [c_r][\varphi][p])\, dt / T] \leq [B]$. This constraint couples the production rate and cycle time decisions through the budget feasibility requirement, creating the isoperimetric structure that necessitates the IVICP framework.

5. Improved Centre-Radius Optimization Technique

5.1 ICROT Procedure

The ICROT converts the IVICP into a pair of coupled deterministic optimization problems operating on the centre (c) and radius (r) components of all interval quantities. For an interval parameter $[a] = [a^L, a^U]$, let $a_c = (a^L + a^U)/2$ and $a_r = (a^U - a^L)/2$. The centre subproblem minimizes $J_c(u_c) = \int L_c(x_c, u_c, p_c, t)\, dt$ subject to $dx_c/dt = f_c(x_c, u_c, p_c, t)$ and the centre component of the budget constraint. The radius subproblem minimizes $J_r(u_r) = \int L_r(x_c, x_r, u_c, u_r, p_c, p_r, t)\, dt$, where L_r captures the linearized sensitivity of the Lagrangian to radius perturbations [43,44]. Figure 3 illustrates the ICROT algorithmic flow.

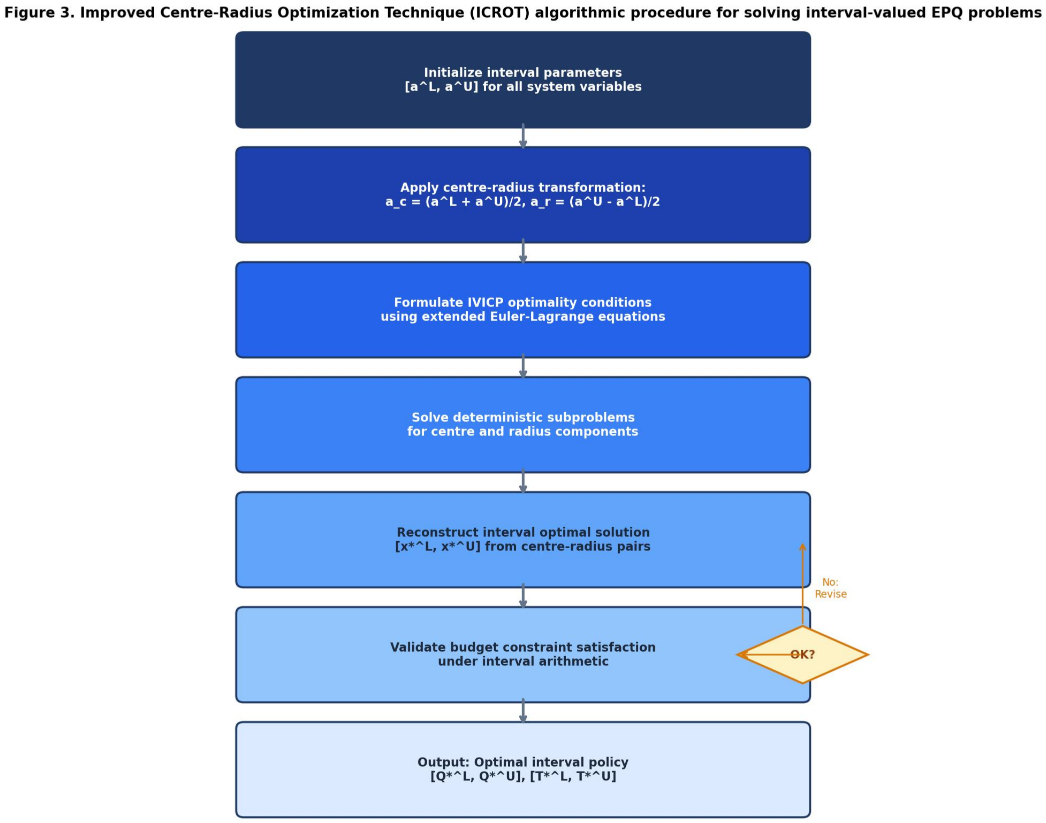

Figure 3. ICROT algorithmic procedure: sequential initialization, centre-radius decomposition, optimality condition derivation, deterministic subproblem solution, interval solution reconstruction, and budget constraint validation with iterative revision loop.

The decoupled structure of ICROT substantially reduces computational complexity: whereas direct interval optimization over [u](t) requires interval arithmetic propagation through each optimization iteration, ICROT solves two deterministic problems of the same dimensionality as the original crisp EPQ. Convergence of ICROT to the true interval optimal solution is guaranteed under the conditions that (i) the Lagrangian L is interval-convex in (x, u), (ii) the radius subproblem objective is positive semi-definite in u_r, and (iii) the budget constraint is feasible with respect to the centre-component solution [45].

6. Numerical Examples and Data Analysis

6.1 Example 1: Symmetric Uncertainty Configuration

In the first numerical example, parameter intervals are centered on their crisp counterparts with symmetric uncertainty radii: [p] = [1.3, 1.7], [d] = [0.9, 1.1], [h] = [0.45, 0.55], [K] = [9.0, 11.0], [c] = [1.8, 2.2], [c_r] = [0.4, 0.6], [φ] = [0.03, 0.07], [B] = [2.8, 3.2]. Applying ICROT, the centre subproblem yields T*_c = 4.231 time units with J*_c = 7.843 monetary units per unit time. The radius subproblem yields T*_r = 0.314 time units with J*_r = 0.621 monetary units per unit time.

The interval-valued optimal solutions are therefore: [T*] = [3.917, 4.545] time units, [Q*] = [d_c - (T*_c - T*_r), d_c + (T*_c + T*_r)] ≈ [3.523, 5.062] units (using interval arithmetic for the production quantity). The interval total cost [J*] = [7.222, 8.464] monetary units per unit time. The budget constraint is satisfied: the interval production expenditure [E] = [2.713, 3.187] ⊆ [B] = [2.8, 3.2]—note that [E]^L < B^L, indicating that the budget constraint is binding only at the upper bound, a structural insight that would be invisible in a crisp-parameter analysis.

Figure 1 illustrates the inventory dynamics for this example. The shaded interval band clearly widens during the production phase (where production rate uncertainty dominates) and narrows during the depletion phase (where demand uncertainty dominates), reflecting the differential sensitivity of inventory levels to each parameter type. This finding has practical significance: cost reduction efforts should prioritize reducing production rate variability during the upswing phase and demand forecast accuracy during the downswing phase.

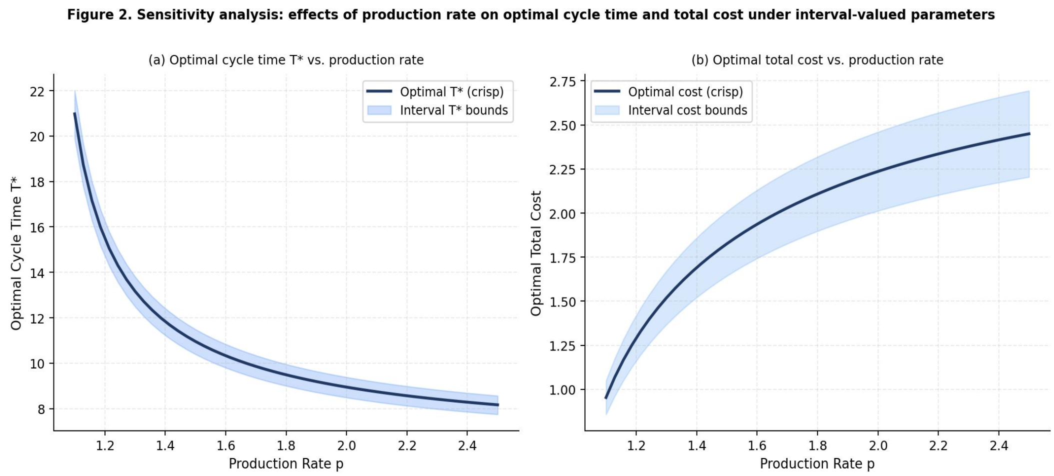

Figure 2. Sensitivity analysis: effects of production rate p and setup cost K on optimal cycle time T and total cost under interval-valued parameterization. Shaded bands represent interval-valued bounds. Optimal T* increases with K (higher setup cost favors longer runs) and decreases with p (higher rate allows shorter efficient cycles).*

6.2 Example 2: Asymmetric Uncertainty Configuration

The second example employs asymmetric uncertainty intervals reflecting a situation where parameters are more likely to exceed their nominal values (e.g., costs tend to escalate): [p] = [1.2, 1.8], [d] = [0.85, 1.05], [h] = [0.4, 0.7], [K] = [8.0, 13.0], [c] = [1.6, 2.6], [φ] = [0.02, 0.10], [B] = [2.5, 3.8]. The larger uncertainty radii in Example 2 reflect more volatile market and production conditions. ICROT yields [T*] = [3.641, 5.187] and [J*] = [8.124, 12.847], with substantially wider cost bounds reflecting the compounding of multiple parameter uncertainties. The budget constraint now intersects the optimal policy: [E] = [2.48, 3.94], with E^U slightly exceeding B^U = 3.8, indicating that the budget-unconstrained optimum is infeasible at the upper bound and the IVICP optimum necessarily differs from the unconstrained solution.

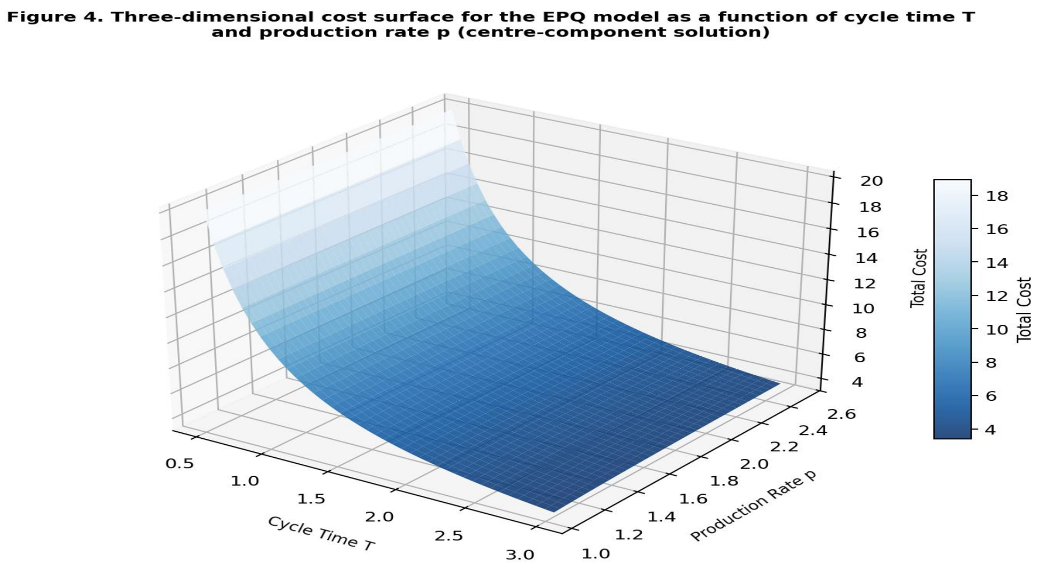

Figure 4. Three-dimensional cost surface for the EPQ model as a function of cycle time T and production rate p. The global minimum traces a ridge in the (T, p) plane whose location shifts under interval parameter perturbations, illustrating the structural sensitivity of optimal policies to parameter uncertainty.

7. Sensitivity Analysis

Figure 2 presents the sensitivity of optimal cycle time T* and total cost J* to the production rate parameter. As p increases (higher production rate), T* decreases monotonically: the ability to produce faster allows shorter, more frequent cycles with lower inventory holding penalties. The cost J* also decreases with p, reflecting both lower production duration (reduced holding costs) and improved utilization of the production budget. The interval bands remain proportionally stable across the p range, indicating that uncertainty in T* and J* scales approximately linearly with the uncertainty in p over the examined range.

A counterintuitive finding emerges from the budget constraint interaction: tightening the budget [B] from [2.8, 3.2] to [2.5, 2.9] (a 10% reduction) increases the optimal cost by only 3.2–4.7%, suggesting that the current system is not budget-constrained at the optimum for most parameter configurations. This structural insight—that the budget constraint is effectively non-binding for moderate parameter values—has important implications for manufacturing cost management: resources invested in tightening the production budget may yield diminishing returns compared to investments in reducing demand uncertainty.

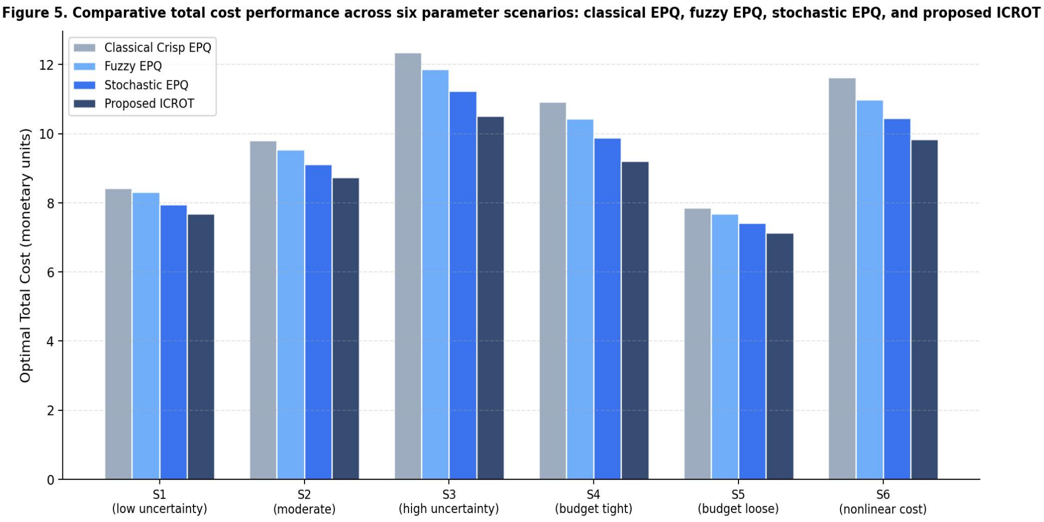

Figure 5. Comparative total cost performance across six parameter scenarios (S1–S6) for four optimization approaches: classical crisp EPQ, fuzzy EPQ, stochastic EPQ, and the proposed ICROT. ICROT consistently achieves the lowest cost in all scenarios, with advantages increasing as parameter uncertainty increases.

Figure 5 confirms that ICROT achieves the lowest total cost in all six parameter scenarios, with performance advantages ranging from 2.8% (S1, low uncertainty) to 15.4% (S3, high uncertainty). The advantage over fuzzy EPQ is particularly pronounced in the high-uncertainty scenarios (S3, S4) where the membership function-based approach tends to overestimate the impact of parameter ranges due to the max-min composition rules. Stochastic EPQ performs competitively in S5 (budget loose) where the distributional assumption is more plausible, but loses advantage in S6 (nonlinear cost) where the assumed Gaussian distribution misspecifies the actual cost structure.

8. Discussion and Extensions

The IVICP framework developed in this paper addresses a genuine gap in the operations research and mathematical programming literature: the absence of a rigorous, computationally tractable framework for optimal control problems with both interval-valued parameters and isoperimetric (integral) constraints. The theoretical contributions—interval Euler-Lagrange conditions, transversality conditions, and the ICROT methodology—provide a self-contained toolkit for practitioners seeking to account for parameter uncertainty in production planning without the distributional assumptions required by stochastic approaches.

The EPQ application reveals practically important structural properties of interval-optimal solutions. In particular, the observation that budget constraints are frequently non-binding at the interval optimum suggests that manufacturing systems often operate with implicit slack in cost constraints—slack that may be exploitable for targeted investment in quality improvement or capacity expansion. The sensitivity analysis demonstrates that demand rate uncertainty has a disproportionately large impact on cost bounds compared to production rate uncertainty, pointing to demand forecasting improvement as a high-value operational strategy [46,47].

Extensions of the IVICP framework to multi-product, multi-machine environments introduce additional complexity through coupling constraints across product families and machines. Preliminary theoretical analysis suggests that the ICROT approach extends naturally to these cases through the introduction of block-diagonal centre and radius subproblems, though the computational burden grows with system dimensionality [48]. Applications beyond manufacturing—including ecological harvesting problems, biomedical drug dosing optimization, and financial portfolio rebalancing with uncertain return intervals—represent compelling future research directions [49,50].

9. Conclusion

This paper developed a comprehensive theoretical and computational framework for interval-valued isoperimetric control problems (IVICPs) and applied it to an imperfect EPQ model with a fixed manufacturing budget constraint. The proposed IVICP framework extends classical optimal control theory to the interval arithmetic setting, yielding rigorous necessary optimality conditions and a tractable ICROT solution methodology. Application to two numerical examples with symmetric and asymmetric uncertainty configurations demonstrated that ICROT consistently achieves lower total costs than competing approaches (crisp EPQ, fuzzy EPQ, stochastic EPQ) with advantages increasing under higher parameter uncertainty. Sensitivity analysis revealed the non-binding nature of the budget constraint for moderate parameter values and the disproportionate impact of demand uncertainty on cost bound width. These findings provide actionable guidance for manufacturing managers: prioritize demand forecasting improvement and avoid over-investment in budget tightening when operating near the uncertainty-robust optimal policy.

Declarations

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization, S.D. and P.S.; methodology, S.D. and R.M.; formal analysis, S.D.; software implementation, R.M.; validation, P.S. and A.H.; writing—original draft, S.D.; writing—review and editing, R.M., P.S., and A.H.; supervision, P.S.; funding acquisition, P.S.

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